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TEMPERATURE DETERMINATION
OF PLANETARY ATMOSPHERES

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CONTRACT NO. NASw-976

PREPARED FOR
NATIONAL AERONAUTICS AND SPACE ADMINISTRATION
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ABSTRACT

A method is presented whereby accurate temperature-altitude profiles of planetary atmospheres may be determined from the number-density profiles of two inert gases having markedly different molecular weights M . In the earth's atmosphere, such gases would preferably be helium and argon. In contrast to previous methods in which mass-density profiles permitted the calculation of only the ratio T/M at altitudes sufficiently below the highest altitude of density data, the two-gas method yields values of kinetic temperature T , not only at low altitudes where number-density data for both gases exist, but also up to the greatest altitude for which the light-gas number-density data have been measured. The method depends upon recently developed mass spectrometers with detection sensitivities of the order of 10^5 particles per cubic centimeter.

A rigorous error analysis predicts the accuracy of the resulting temperatures on the basis of sensor and telemeter characteristics, and allows for optimizing any actual experiment as far as range and number of measurements are concerned.

SUMMARY

A new method is described which allows an accurate determination of the temperature profile in a heterogeneous atmosphere in which the several constituents are in diffusive equilibrium. This method involves the use of a rocket- or satellite-borne mass spectrometer for the measurement of the individual number-density profiles of two atmospheric species of widely differing mass number. These data yield temperature profiles with an accuracy depending only upon the accuracy of the mass spectrometer. Existing equipment permits the determination of the earth's atmospheric temperature with an accuracy equal to the accuracy of the number-density data for altitudes below 250 km.

The first part of this paper deals with some aspects of the development of the temperature-density relationships. Recently, careful studies of this problem have been undertaken in connection with the creation of new model atmospheres. For this reason, this paper mentions only those facts which are closely related to the problem of temperature determination from density profiles.

The innovations which yield the temperature profile are discussed in detail and a hypothetical application of this method to the upper part of the earth's atmosphere, demonstrates the power of this approach.

Numerical calculations were performed as an illustration of this method using the density profiles of two rare gases - He and A - which seem to be very suitable because of their large difference in molecular weight and their chemical inertness.

The second part of this paper consists of an extensive error analysis. Because of the complexity of the problem no direct and reliable estimations of the errors resulting from experimental uncertainties can be made. Therefore, a vigorous mathematic error analysis was performed with two goals in mind: (1) to determine the total uncertainty of the final results and (2) to determine optimum conditions under which the final errors will become minimum.

Considerable thought was given to the problem of how the number of data points used influences the final results; in particular, a study is made of the problem of the statistical treatment of data points so close together in altitude, that they cannot be considered independent.

The influence of sensitivity and noise level of the mass spectrometer and of reading accuracy associated with the telemeter system, were carefully considered. All these investigations together allow one to obtain the important parameters for the proper design of the experiment as far as instrumentation and range of investigation is considered.

The error analysis was performed in a very general way which allows one to adopt this method to many other problems of the evaluation of data obtained by rocket and satellite experiments.

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SECTION 1

INTRODUCTION

Knowledge of the atmospheric temperature profile as a function of height is of great importance for the proper interpretation of many physical effects in any given planetary atmosphere and for the understanding of the mechanism involved. In particular, even small changes in the temperature profile can have considerable effects on the pressure and density distribution in any atmosphere.

The temperature with its altitude-dependent variations is one of the basic atmospheric parameters since it is the defining property of many other physical properties of an atmosphere. Absorption of solar radiation at some altitude, for instance, may result in an increase of temperature which, as a consequence, may change the density of this atmosphere at much higher altitudes by orders of magnitude.

Any direct approach to the measurement of temperature in a rarefied gas is practically impossible, and indirect methods applied to the upper regions of any planetary atmosphere have to date yielded results with rather limited accuracy. On the other hand, any discussion of atmospheric properties must be based on realistic model atmospheres which

cannot be created without reasonable estimates of the temperature distribution with height. In the case of the earth's atmosphere, some model atmospheres (Minzner, 1956; Minzner, 1958; Minzner, 1959; Champion and Minzner, 1963) have been developed from assumed temperature profiles which were adjusted repeatedly until the pressure and density values calculated from these temperature profiles matched the observed rocket and satellite data within limits compatible with the wide spread of data.

With the availability of rocket- or satellite-borne high sensitivity mass spectrometers (G. Sauermann and R. Herzog, 1961), a new method to obtain a temperature profile can be suggested. The generally used approach, to obtain temperatures from density or pressure profiles, implied knowledge of the mean molecular weight of the atmosphere and in particular the change of the mean molecular weight with altitude. Therefore, absolute temperature determinations could not be made accurately, especially at altitudes above 250 km where these determinations may be off by a factor of two.

The method suggested in this paper relies on the knowledge of the number-density profiles of two inert gases such as helium and argon with widely differing atomic weights. The great advantage of this choice is that the distribution of these two gases with height follows the thermodynamical properties of the atmosphere only; no dissociation or ionization effects have to be considered, and it is thought that charge exchange processes do not seem to effect the distribution of these two gases, at least up to altitudes of approximately 1000 km.

It will be shown that basically no initial knowledge of the temperature at any altitude is necessary to establish a temperature profile, as long as both gases are in diffusive equilibrium. In order to examine the suitability of the approach, the method was subjected to a vigorous error analysis which proved the validity of the general principle. Considering the complexity of the problem, the resulting errors can be kept surprisingly small, approximately 10% at 700 km altitude, as a numerical calculation for the case of the earth's atmosphere demonstrated.

This paper represents a modification and condensation of an earlier study published as a report (Minzner, Sauermann and Peterson 1963) under Contract NASw-394.

SECTION 2

THE RELATIONSHIP OF TEMPERATURE TO NUMBER DENSITY

We consider an altitude region in which diffusive separation dominates the distribution of the separate atmospheric gases and in which dissociation, ionization, and chemical reactions involving these gases can be neglected. One may then use the hydrostatic equation and kinetic-theory considerations to obtain the following well-known equation giving temperature T_2 in terms of T_1 and the values of number density versus altitude, $n(h)$, for a particular atmospheric constituent of molecular weight M :

$$T_2 = \frac{n_1}{n_2} T_1 - \frac{GM}{R} \int_{h_1}^{h_2} \frac{n(h)}{n_2} dh \quad , \quad (1)$$

where R is the universal gas constant and where the subscripts 1 and 2 specify the value of n or T for particular altitudes consistent with the limits of the integral. The results are expressed in terms of geopotential altitude " h " (V. Bjerknes, 1910; L.P. Harrison, 1951) by means of the defining transformation $g(z)dz = Gdh$, where $g(z)$ is the variable acceleration of gravity at geometric altitude " z " and where G is a constant scale factor. The use of geopotential altitude eliminates the need to

account further for changes in the acceleration of gravity with altitude.

In the form when h_1 is greater than h_2 , Equation (1) has been used by Elterman (1953) to deduce temperature from total number-density data at altitudes below the reference level h_1 for which an initial temperature T_1 must be assumed or determined independently. For this application M was considered to be the mean molecular weight.

As has long been realized, Equation (1) has a practical limitation which makes it almost useless for certain situations. The difficulty comes in computing temperatures T_2 at altitudes considerably above the reference altitude, for which case $n_2 \ll n_1$. For such a calculation one finds that the values of T_2 depend on relatively small differences between two nearly identical quantities and therefore are of dubious value, especially since one of the quantities is equal to T_1 times a factor $n_1/n_2 \gg 1$, which correspondingly magnifies any uncertainty inherent in T_1 . Although this effect is a serious limitation for a heavy gas like argon, one finds that, for a light gas like helium, the number-density gradient with respect to altitude, in the region of diffusive separation, is small enough that the undesirable condition $n_2 \ll n_1$ is not reached except for a very large altitude increment, i.e., of the order of 1000 km.

Equation (1) applied in a downward direction, on the other hand, has no such limitation. Now, in fact, since $n_2 > n_1$, one finds that the further the computation is carried below the reference level h_1 , the less important is the term containing the reference temperature T_1 . This

favorable situation is enhanced when the number-density gradient is large, as for a heavy gas in the region of diffusive separation. In summary, then, a heavy gas is preferable for a downward-proceeding calculation; a light gas is preferable for an upward-proceeding calculation.

The two equations implied above will be referred to as the heavy-gas-down equation and the light-gas-up equation, and are stated as

$$T_b = \frac{n_a^*}{n_b^*} T_a + \frac{GM^*}{R} \int_{h_b}^{h_a} \frac{n^*(h)}{n_b^*} dh, \quad (2)$$

$$T_a = \frac{n_b}{n_a} T_b - \frac{GM}{R} \int_{h_b}^{h_a} \frac{n(h)}{n_a} dh. \quad (3)$$

In these two expressions, as in all later work, the subscript "a" refers to the higher altitude and "b" to the lower altitude. In addition, the heavy-gas number densities and the heavy-gas molecular weight are identified by asterisks, while the light-gas number densities and molecular weight are designated by regular type.

Thus, to determine the temperature profile for an altitude region, it would seem desirable to obtain the number densities for two independent neutral gases - one a heavy gas, and the other a light gas. The heavy-gas number densities $n^*(h)$ would first be used in the downward calculation, after assuming some reasonable initial reference temperature T_{a_0} . The temperature T_{b_1} , computed at the lower altitude from the heavy-

gas-down calculation, would then serve as the reference temperature assumed for an upward calculation using the light-gas number densities $n(h)$. The process may be used iteratively to improve successively the computed values of temperature between altitudes a and b . After N iterations, for example, one obtains

$$T_{b_N} = \left(\frac{n_a^*}{n_b^*} \frac{n_b}{n_a} \right)^N \frac{n_a}{n_b} T_{a_0} + A \left[1 + \frac{n_a^*}{n_b^*} \frac{n_b}{n_a} + \dots + \left(\frac{n_a^*}{n_b^*} \frac{n_b}{n_a} \right)^{N-1} \right] \quad (4)$$

$$- B \frac{n_a^*}{n_b^*} \left[1 + \frac{n_a^*}{n_b^*} \frac{n_b}{n_a} + \dots + \left(\frac{n_a^*}{n_b^*} \frac{n_b}{n_a} \right)^{N-2} \right] ,$$

$$T_{a_N} = \left(\frac{n_a^*}{n_b^*} \frac{n_b}{n_a} \right)^N T_{a_0} + A \frac{n_b}{n_a} \left[1 + \frac{n_a^*}{n_b^*} \frac{n_b}{n_a} + \dots + \left(\frac{n_a^*}{n_b^*} \frac{n_b}{n_a} \right)^{N-1} \right] \quad (5)$$

$$- B \left[1 + \frac{n_a^*}{n_b^*} \frac{n_b}{n_a} + \dots + \left(\frac{n_a^*}{n_b^*} \frac{n_b}{n_a} \right)^{N-1} \right] ,$$

where

$$A = \frac{GM^*}{R} \int_{h_b}^{h_a} \frac{n^*(h)}{n_b^*} dh \quad ; \quad B = \frac{GM}{R} \int_{h_b}^{h_a} \frac{n(h)}{n_a} dh .$$

Under the conditions that $\frac{n_a^*}{n_b^*} \frac{n_b}{n_a} < 1$ and N is a large number, Expressions (4) and (5) converge exactly to the equations

$$T_b = \frac{1}{\left(\frac{n_b^*}{n_a^*} - \frac{n_b}{n_a} \right)} \left[\frac{GM^*}{R} \int_{h_b}^{h_a} \frac{n^*(h)}{n_a^*} dh - \frac{GM}{R} \int_{h_b}^{h_a} \frac{n(h)}{n_a} dh \right] , \quad (6)$$

$$T_a = \frac{1}{\left(\frac{n_a}{n_b} - \frac{n_a^*}{n_b^*} \right)} \left[\frac{GM^*}{R} \int_{h_b}^{h_a} \frac{n^*(h)}{n_b^*} dh - \frac{GM}{R} \int_{h_b}^{h_a} \frac{n(h)}{n_b} dh \right] . \quad (7)$$

Expressions (6) and (7) will be referred to as the double-gas-down equation and the double-gas-up equation, respectively.

For many conditions the interval a to b and the differences in molecular weight of the two gases are such that the ratio $\frac{n_a^*}{n_b^*}$ is several orders of magnitude smaller than $\frac{n_a}{n_b}$. In this case only one iteration is needed [$N = 1$ in Equations (4) and (5)] to arrive at the essentially correct temperatures T_a and T_b . This is seen from the fact that the heavy-gas-down expression, under the above restrictions, gives the correct temperature T_b to within very close approximation and, therefore, when used in the light-gas-up expression gives the correct temperature profile from b to a . It is expected, from these arguments, that if $\frac{n_a^*}{n_b^*} \ll \frac{n_a}{n_b}$, the two double-gas equations must degenerate to the single-gas equations, at least near the end points a and b . It is indeed found that Equations (6) and (7) after slight simplification become

$$T_b = \left(T_a - \frac{n_b}{n_a} T_b \right) \frac{n_a^*}{n_b^*} + \frac{GM^*}{R} \int_{h_b}^{h_a} \frac{n^*(h)}{n_b^*} dh \quad (6a)$$

$$T_a = \left(T_b - \frac{n_a^*}{n_b^*} T_a \right) \frac{n_b}{n_a} - \frac{GM}{R} \int_{h_b}^{h_a} \frac{n(h)}{n_a} dh \quad (7a)$$

These expressions obviously approach the heavy-gas-down and the light-gas-up equations when $\frac{n_a^*}{n_b^*} \ll \frac{n_a}{n_b}$.

The double-gas equations have the rather pleasant feature that the temperature T_a or T_b can be computed directly without ever assuming a reference temperature. If the number densities or the relative number densities for two different gases are known exactly, then the temperature profile can be determined exactly, and is unique. If, on the other hand, the same data are known exactly for only one gas, the temperature profile can be determined uniquely only if the exact temperature at some reference point within the interval is known. Whether we use the double-gas equations or the single-gas equations (assuming the exact reference temperature to be known), the temperature profile is, of course, the same.

The double-gas equations can also be obtained by solving Equations (2) and (3) simultaneously, assuming that T_a and T_b in one equation have the same meaning as in the other equation. The derivation by the iteration procedure does, however, aid considerably in understanding the relationships of the various equations.

SECTION 3

NUMERICAL ILLUSTRATION

The ideas contained in the preceding section are illustrated numerically by means of a hypothetical set of helium and argon number densities shown in Figure 1. These densities are computed from the standard atmosphere temperature profile (Champion et al, 1962; Champion and Minzner, 1963) on the basis of a micropause level at 120 km.* According to this model, argon number-density values vary more than a factor of 10^5 between 150 and 450 km altitude, the approximate point where argon signal-to-noise ratio falls below unity in a mass spectrometer having a sensitivity of 10^{10} particles per meter³. Helium, on the other hand, varies by less than a factor of 10 between 150 and 700 km, an altitude where the number density is still considerably above the noise level of a mass spectrometer having the sensitivity mentioned above.

Since the temperature profile and the number densities used in the illustrations are self consistent, the integrals arising from Equation (1) in its various forms can be evaluated indirectly by means of the relationship

* Throughout this paper geometric altitude is identified by the units km and geopotential altitude by the units km'.

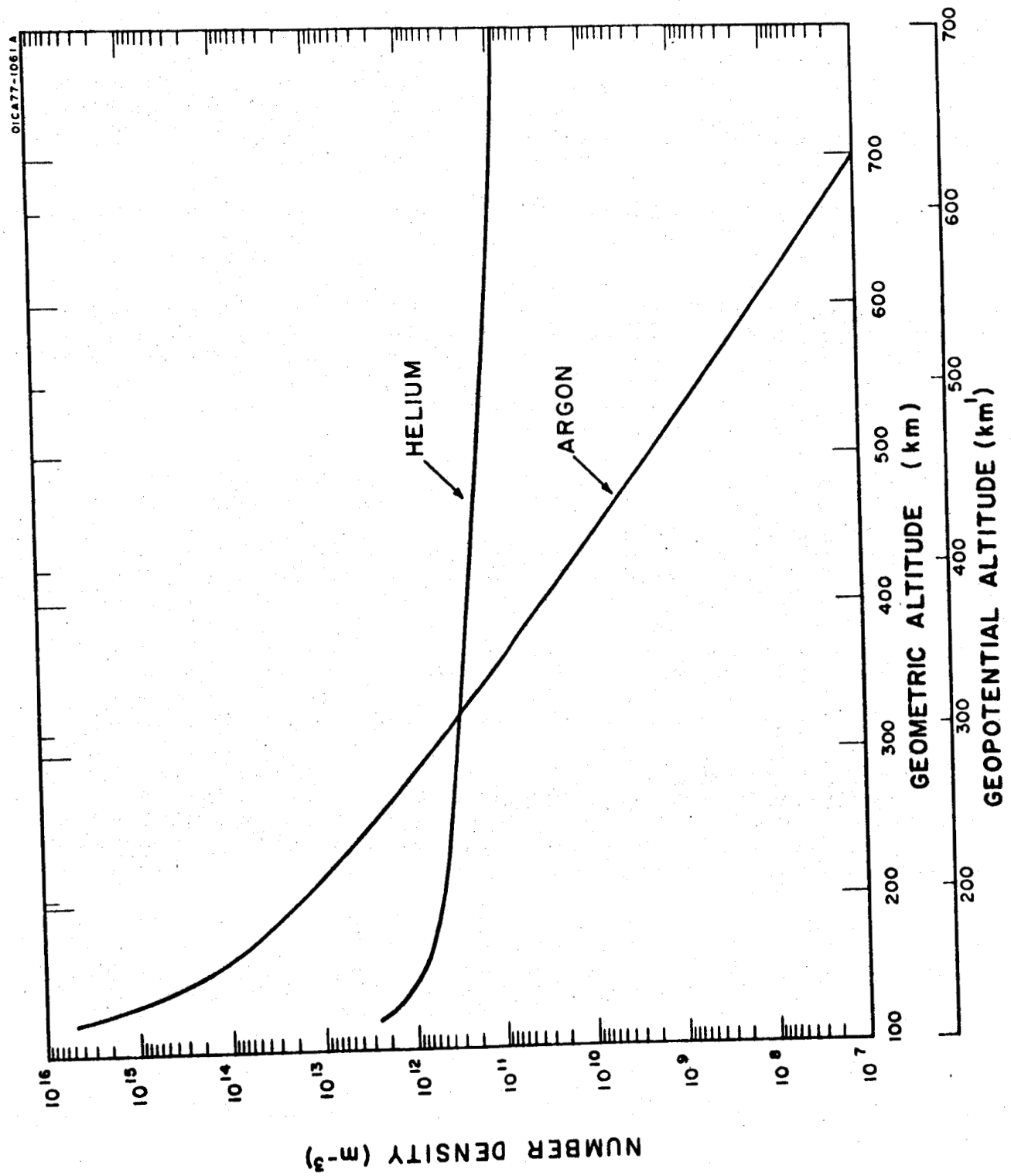


Figure 1. Number density of helium and argon vs altitude.

$$\frac{GM}{R} \int_{h_1}^{h_2} \frac{n(h)}{n_2} dh = \frac{n_1}{n_2} T_1 - T_2 \quad (8)$$

Of course, when doing an actual experiment, the temperature profile is the unknown to be found; thus, the integral must be evaluated directly by numerical methods. For example, if q heavy-gas density-data points are available, one might use the following form of Equation (2) obtained by applying the trapezoidal rule to evaluate the integral:

$$T_b = \frac{n_a^*}{n_b^*} T_a + \frac{GM}{Rn_b^*} \sum_{j=2}^{q-1} n_j^* \Delta h_j + \frac{GM}{Rn_b^*} \frac{(n_a^* + n_b^*) \Delta h}{2} \quad (2a)$$

Figure 2 shows the relative importance of the two terms of the general single-gas-down equation resulting when $h_2 < h_1$ in Equation (1). Because the equation is defined in terms of geopotential altitude, Figure 2 is also in terms of geopotential. The heavy line in Figure 2 is the correct temperature T and the lower dashed lines represent the values of the integral term in Equation (1) when applied to argon and helium. The values of the ratio term $\frac{n_1}{n_2} T_1$ are represented by the differences between the temperature line and the dashed lines at any altitude h_2 . For argon, the value of this term drops to a negligibly small value (less than 1%) at altitudes below $h_2 = 300$ km' when the reference level h_1 is 450 km'. For helium, on the other hand, with the reference level h_1 at 700 km', the ratio term never becomes negligibly small within the region of interest,

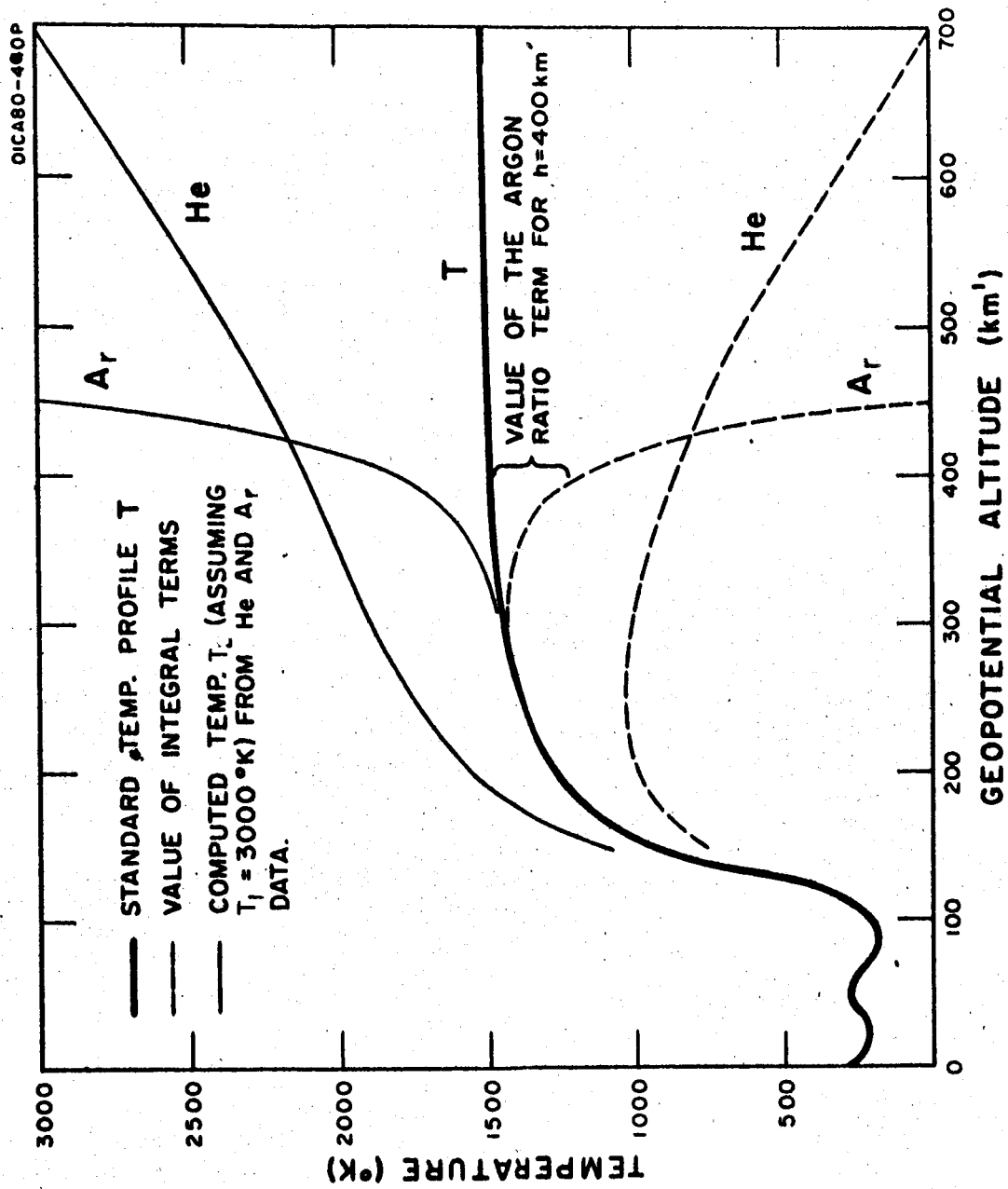


Figure 2. Contribution of the two terms of the single-gas-down temperature equation for both helium and argon number densities.

and is in fact still 18% of T at $h_2 = 150 \text{ km'}$. An altitude of 450 km' was chosen as an upper limit for plotting Equation (1) in terms of argon data since this is the altitude above which argon can no longer be detected by any presently available mass spectrometers. For helium an upper altitude limit of 700 km' was chosen, although one could almost certainly go much higher.

If T_1 is incorrectly assumed to be 0°K for an argon calculation, for example, the value of the ratio term is zero. The computed values of T_2 would thus follow the dashed line labeled Ar, and accurate values of T_2 would be obtained at altitudes below 300 km' . If, on the other hand, T_1 is assumed to be 3000°K in an argon calculation, the value of the ratio term is 3000° at 450 km' and the computed values of T_2 follow the solid line labeled Ar. Again, accurate values of T_2 are obtained below 300 km' . For a similar range of assumed values of T_1 with helium number-density data, however, no realistic values of T_2 are forthcoming for any part of the altitude region where diffusive separation may exist (i.e., above 120 km'). The single-gas-down equation is obviously unsuitable for a gas as light as helium unless reliable high-altitude reference temperatures are available.

Figure 3 presents a similar evaluation of the general single-gas-up equation obtained when $h_2 > h_1$ in Equation (1). A semi-log plot is made necessary by the large range of values, with the heavy line again being temperature T . The dashed line, representing the value of the

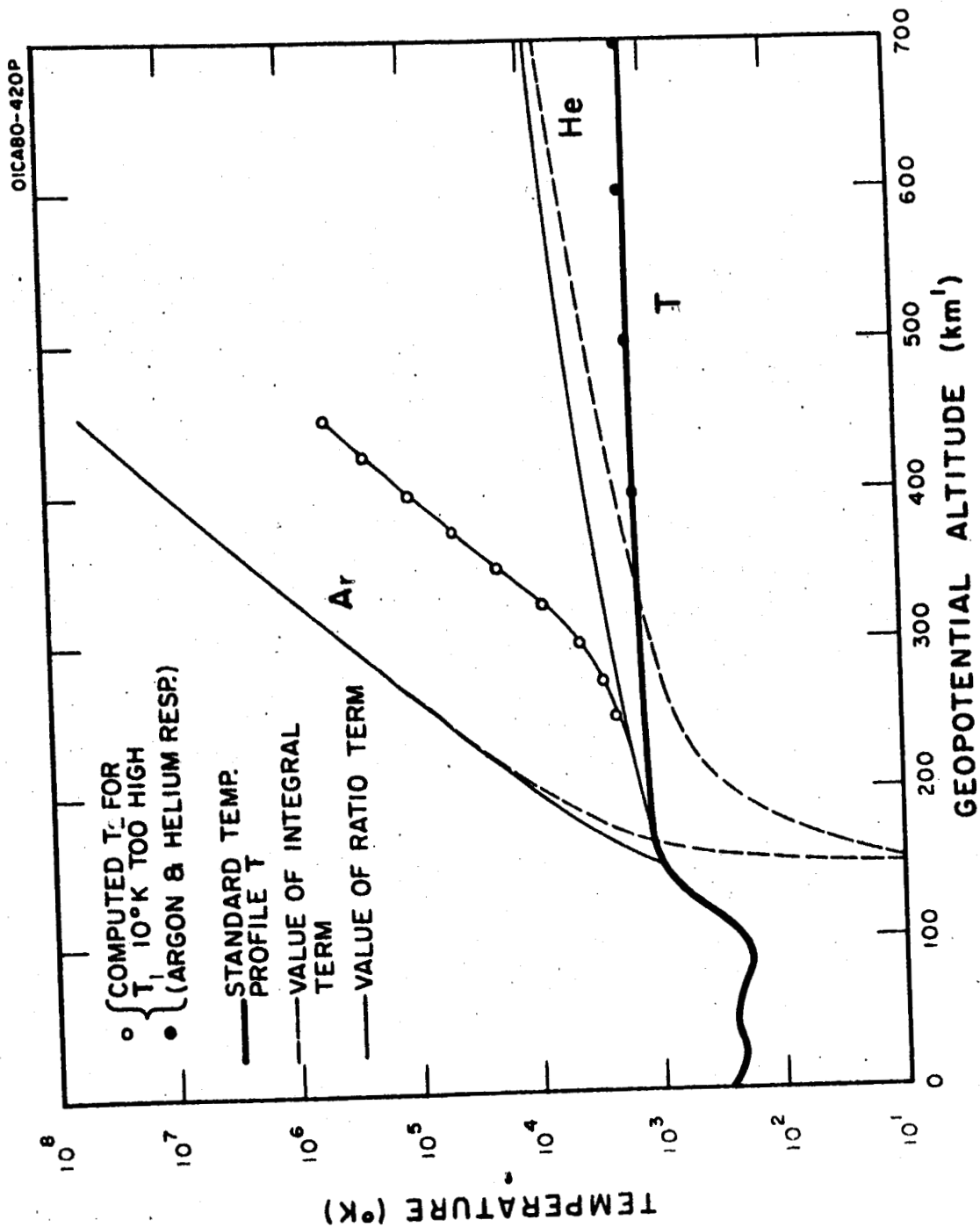


Figure 3. Contribution of the two terms of the single-gas-up temperature equation for both helium and argon number densities.

argon integral term, increases indefinitely with increasing altitude as does the argon ratio term represented by the light solid line labeled Ar. The temperature T is the difference between the two terms represented by these two lines. Such a relatively small difference (of the order of 10^3) between two very large values (i.e., 10^8 at 450 km') is very inaccurate. An uncertainty of 10°C in T_1 at 150 km' propagates to an error of 10^6 degrees C at 450 km'. It is apparent that the single-gas-up equation is not suitable for a gas as heavy as argon. With helium, however, the ratio term and the integral term each have a value of the same order as the temperature. In this instance, an error of 10°C at 150 km' propagates to an error of only 100°C at 700 km', and it appears that this equation is quite suitable for a light gas calculation, provided of course that enough gas is present for the measurement.

Figure 4 shows the evaluation of the double-gas-down expression as given by Equation (6), presented on a semi-log plot. The heavy line represents the temperature T , the two dashed lines are the values of the argon and helium integrals, and the light solid line represents the value of the dimensionless ratio factor. The difference between the integral terms multiplied by the ratio factor yields T_b . In this form, the relationship of the double-gas-down to the heavy-gas-down equation is not evident. If, however, each integral term is separately multiplied by the ratio factor the resulting values are the open circles for the argon term and the solid circles for the helium term. As b moves below 300 km', it is evident that the contribution from the helium term becomes small

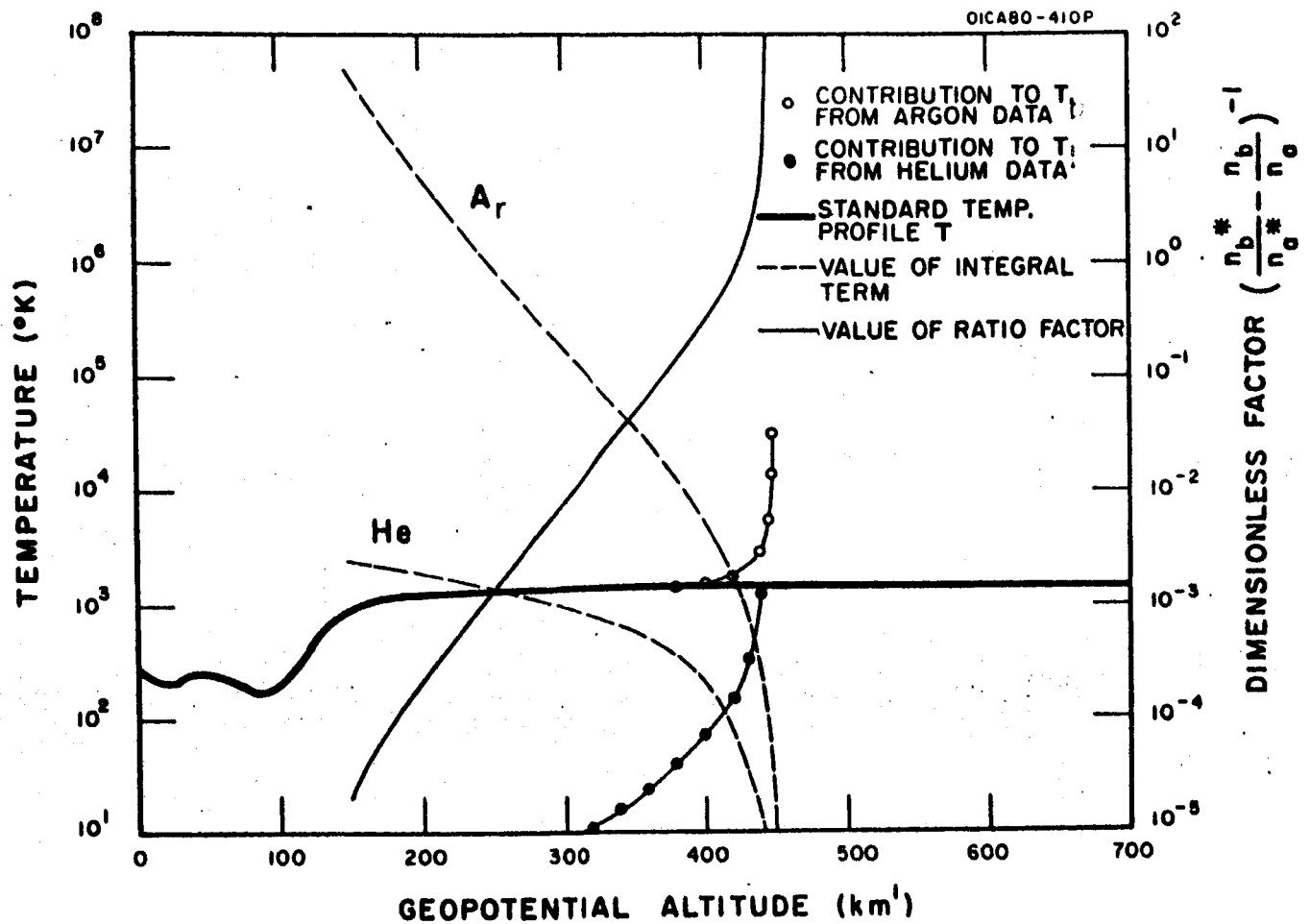


Figure 5. Contribution of the two integral terms and the ratio factor of the double-gas-down temperature equation for helium and argon number densities.

and the double-gas-down equation becomes essentially equal to the heavy-gas-down equation at low altitude.

Finally, in Figure 5 is illustrated the double-gas-up expression, as given by Equation (7). As in Figure 4 the heavy line represents the temperature T , the two dashed lines represent the integral terms, and the light solid line represents the ratio factor. The value of the argon integral term is seen to rise rapidly until it approaches a constant near 300 km'. The value of the helium integral term continues to rise with altitude but always remains below the value of the argon integral. Unlike the three previous cases, it is not readily demonstrated graphically that the double-gas-up equation approaches the light gas up equation, at least with the form of the double-gas-up equation as presented.

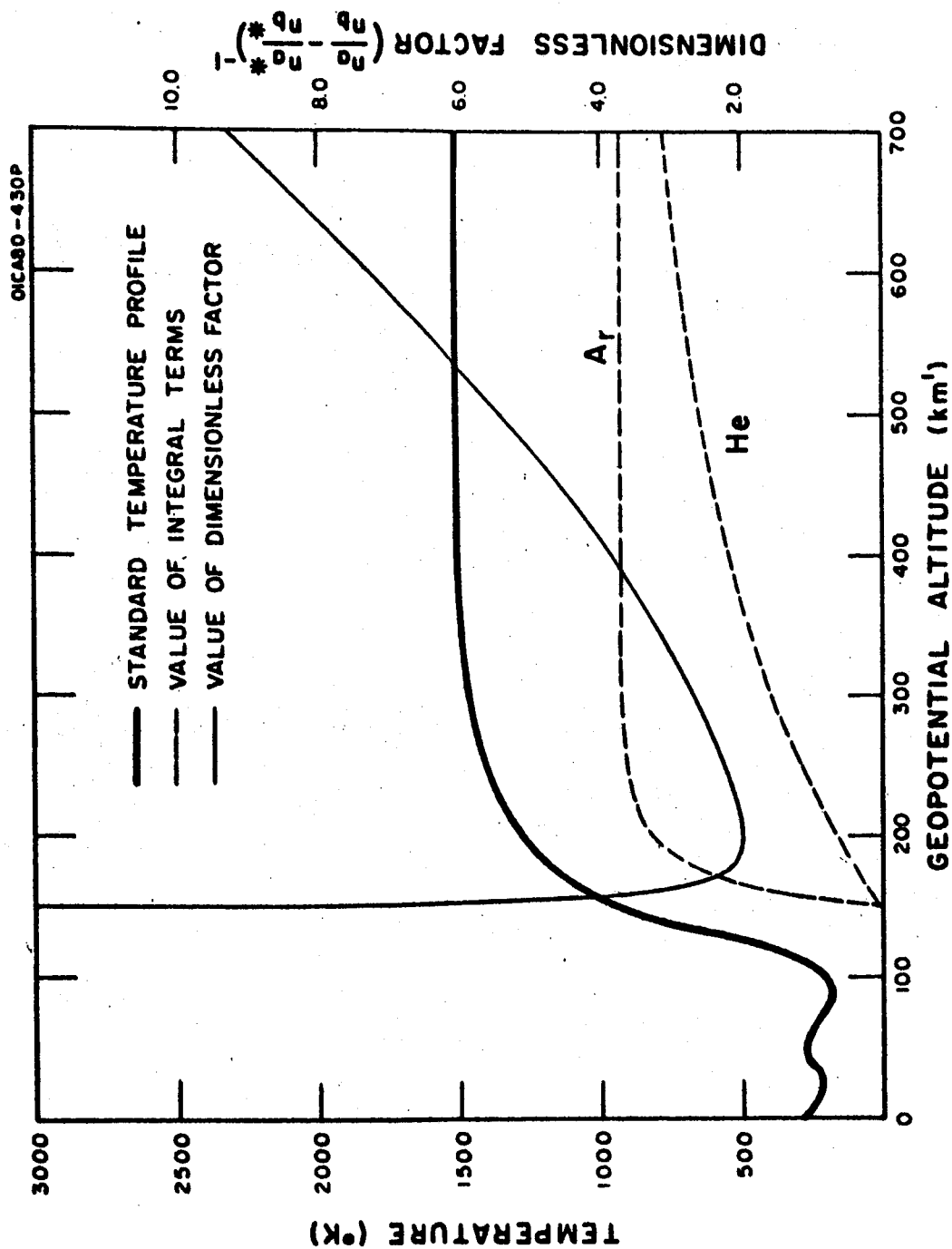


Figure 4. Contribution of the two integral terms and the ratio factor of the double-gas-up temperature equation for helium and argon number densities.

SECTION 4

ERROR ANALYSIS - GENERAL

Let us assume that number-density data for two or more gases in the atmosphere are given. The uncertainties associated with each temperature equation are likely to vary widely, and each equation will have regions of optimum utility. To investigate the behavior of the uncertainties the gaussian method has been used, wherein each data point is assumed to be normally distributed with an average value y and having a variance δy . Thus, if x is a function of the independent variables y_i and if each variable y_i has an uncertainty δy_i , then the uncertainty δx in x is approximated by

$$(\delta x)^2 = \sum_i \left(\frac{\partial x}{\partial y_i} \delta y_i \right)^2 \quad (9)$$

As a first example of the method, let us assume that the temperatures are computed by considering each density point to be independent in the sense that the value of one point tells nothing about the value of a neighboring point. By Equation (9) then, (after assuming that the integrals are evaluated numerically) one obtains a set of expressions

for the temperature uncertainties, one expression for each of the four temperature equations discussed previously. In order to distinguish between the various temperature uncertainties, a special notation is used, as explained by the following examples: $(\delta T \frac{a}{b})_h$ signifies the uncertainty in temperature at altitude h_b as determined from the heavy-gas-down equation and computed from the reference level h_a . Similarly $(\delta T \frac{a}{b})_d$ identifies the uncertainty at altitude h_a as obtained from the double-gas-up equation with h_b as the reference level. The underline below "a" or "b" denotes the reference level, and the subscript "h", "l" or "d" denotes the use of the heavy-gas, light-gas, or double-gas equation. The set of temperature uncertainties is as follows:

$$(\delta T \frac{a}{b})_h = \frac{1}{n_b^*} [u^2 + (n_a^* \delta T_a)^2]^{1/2} , \quad (10)$$

$$(\delta T \frac{a}{b})_l = \frac{1}{n_a} [v^2 + (n_b \delta T_b)^2]^{1/2} , \quad (11)$$

$$(\delta T \frac{a}{b})_d = \frac{1}{\left(\frac{n_a}{n_b} - \frac{n_a^*}{n_b^*}\right)} \left[\left(\frac{u}{n_b^*}\right)^2 + \left(\frac{v}{n_b}\right)^2 \right]^{1/2} , \quad (12)$$

$$(\delta T \frac{a}{b})_d = \frac{1}{\left(\frac{n_b^*}{n_a^*} - \frac{n_b}{n_a}\right)} \left[\left(\frac{u}{n_a^*}\right)^2 + \left(\frac{v}{n_a}\right)^2 \right]^{1/2} , \quad (13)$$

where, for example, if the necessary integrals have been evaluated by

the trapezoidal rule using equal intervals Δh one finds that

$$u^2 = (T_a + \frac{GM^*}{R} \frac{\Delta h}{2})^2 (\delta n_a^*)^2 + (T_b - \frac{GM^*}{R} \frac{\Delta h}{2})^2 (\delta n_b^*)^2 + (\frac{GM^*}{R})^2 \sum_{j=2}^{q-1} (\delta n_j^*; \Delta h)^2, \quad (14)$$

$$v^2 = (T_a + \frac{GM}{R} \frac{\Delta h}{2})^2 (\delta n_a)^2 + (T_b - \frac{GM}{R} \frac{\Delta h}{2})^2 (\delta n_b)^2 + (\frac{GM}{R})^2 \sum_{j=2}^{q-1} (\delta n_j; \Delta h)^2. \quad (15)$$

The uncertainties in number density δn at various altitudes and for various gases would normally be related to the standard deviations of the measurements. These uncertainties depend upon the precision of the particular measuring instruments, i.e., upon the reading error associated with the output presentation or associated with the telemetry system, and upon the ultimate sensitivity which determines the limitations in the region of low signal-to-noise ratio.

A second type of uncertainty, not associated directly with statistics but nevertheless of vital concern, is the calibration accuracy. This calibration accuracy, except for non-linearity and the usually negligible short-term drift associated with transient phenomena, does not enter into the calculations of temperature since number densities always appear as ratios in which the calibration constant converting output current to number density cancels. The absolute number densities are necessary, however, in order to provide information on the temperature uncertainties, which contain not only ratios like $\frac{n_a}{n_b}$ but also terms like $\frac{\delta n_b}{n_b}$. It is

obviously of importance to have a calibration uncertainty small with respect to the truly random errors.

It is difficult to deduce the uncertainties in number density for many data obtained from rocket flights, (largely for lack of sufficient concern by the observer during design, calibration and reduction of the data) but one can generally guess what the order of magnitude of these uncertainties might be. Let us assume that the measuring instrument is a mass spectrometer having a sensitivity Δn , and that the combined effects of reading error, telemetry, non-linearity and errors in aerodynamic transformations are expressible as a fractional uncertainty Δt . Then, as an order-of-magnitude estimate of δn , for the spectrometer under ambient conditions, one might consider the function

$$\frac{\delta n}{n} = \frac{\Delta n}{n} + \Delta t \quad (n > \Delta n) . \quad (16)$$

This function reproduces the general features expected, i.e., when n is much greater than the noise level, $\delta n/n$ is dominated by overall reading errors, and when n approaches the noise level $\delta n/n$ approaches unity. It is recognized that the term $\frac{\Delta n}{n}$ tends to be pessimistic, especially when the noise level is a relatively well-behaved, predictable quantity. Therefore, under some circumstances, it might be better to replace $\frac{\Delta n}{n}$ by $f(\Delta n) \frac{\Delta n}{n}$ where $f(\Delta n)$ is some proportionality factor, with this factor dependent on the particular instrumental characteristics near the noise level.

The uncertainty in temperature, δT_a or δT_b , at the reference altitude

presents a different type of problem from that of the uncertainty in number-density. Since temperature is the unknown and is not directly measured, the uncertainty in temperature at the reference point must be either guessed or else determined by an independent method. The single-gas equations therefore have obvious limitations.

If one has number-density data for two neutral gases, and if nothing is known about the temperature, the following general procedure would be followed. First, the altitude region in which the data from the two gases overlap is used. In this region the temperature and the temperature uncertainty is determined from the double-gas equations, always choosing at each altitude the temperature equation that gives the lower uncertainty. With the endpoint values of the double-gas region as references, one then uses the single-gas equations to extend the temperature and temperature uncertainty above the upper endpoint and below the lower endpoint. If, however the temperature at a point is known to within some close limits, it might be profitable to use these limits and work with the single-gas equations. Examples of how the data can be used in various ways to obtain the lowest uncertainties will be shown in Section 5 of this paper where numerical evaluations of the error equations are carried through.

It was seen in an earlier section that for a large enough interval, only one iteration of the two single-gas equations was needed to give the essentially correct temperature profile. The uncertainty equations have

a similar behavior although this behavior is not as well pronounced as it is in the temperature equations and cannot be stated as a general result. It can be demonstrated that under almost any physically expected conditions provided $\frac{n_a^*}{n_b} \ll \frac{n_a}{n_b}$, Equations (10) and (13) converge to the common value

$$(\delta T \frac{a}{b})_h \approx (\delta T \frac{a}{b})_d \approx \frac{u}{n_b} \quad (17)$$

for any reasonable input uncertainty δT_a assumed for Equation (10). This is certainly true for the helium-argon data shown in Figure 1. When this new uncertainty is now used in the light-gas-up uncertainty, Equation (11), one finds that the constant value of u from Equation (17) is approached asymptotically by the variable u which appears in Equation (12), the double-gas-up uncertainty equation. Therefore the light-gas-up and the double-gas-up uncertainties converge if Equation (17) is the input for the light-gas-up equation.

$$(\delta T \frac{a}{b})_l \approx (T \frac{a}{b})_d \approx \frac{n_b}{n_a} [(\frac{u}{n_b})^2 + (\frac{v}{n_b})^2]^{1/2} \quad (18)$$

It is important to realize, however, that formal iteration of the single-gas uncertainty equations will not lead to the double-gas uncertainty equations. It is in fact incorrect to iterate the uncertainty equations because the data would then in effect be used more than once, and this is not consistent with the basic ideas of statistics.

Up to now the equations have assumed the complete independence of

each data point from every other point. Let us observe what effect this has on the results. As the number of data points increases indefinitely, the intervals Δh in Equations (14) and (15) become smaller and smaller, leading ultimately to the uncertainty expressions defined by the conditions

$$u^2 = T_a^2 (\delta n_a^*)^2 + T_b^2 (\delta n_b^*)^2 , \quad (19)$$

$$v^2 = T_a^2 (\delta n_a)^2 + T_b^2 (\delta n_b)^2 . \quad (20)$$

This, of course, implies the illogical result that there is a limit beyond which more data do not help to reduce the uncertainty in temperature. Therefore it seems that a point-by-point treatment of the data is not always the best technique and certainly becomes progressively worse as the data points get closer together. Physical reasoning tells one, that if two or more data points lie very close to each other in altitude, then knowledge of one point gives information about the neighbors simply because the temperature and density of the atmosphere are slowly varying quantities. If the neighboring points are far apart, knowledge of one point does not contribute much to the knowledge of its neighbors, and only then does the assumption, that each data point is independent, become meaningful. A useful and reasonable empirical criterion for ensuring that point-by-point treatment of the data is as good as one can do is that the true difference between the actual values of two neighboring number-densities be several times larger than the uncertainty in each point.

The correct way to handle the small Δh problem, i.e., when the differences between adjacent points is small compared with the uncertainty is to use some type of least-squares fit to the data and then proceed from there. This is often a difficult and always a tedious ordeal, however. One alternate approach, which is easy to apply and is simple in concept, is to average two, three, or more adjoining points using each point only once. The decision on how many points to average would be made on the basis of how close the points are in altitude coupled with what is the uncertainty in each point. As an example, suppose one first decides to average pairs of data points. The average value would then apply to the mid altitude of the adjoining points, and the uncertainty of the average value would be $1/\sqrt{2}$ times the average uncertainty of the two points. If the new values with their new lower errors satisfied the conditions that neighboring points differ in magnitude significantly compared with their uncertainties, then these new points may be used in the temperature expressions, Equations (2), (3), (6), (7) and in the error expressions, Equations (10) through (13); if not, then three (or more) adjoining points can be averaged until one reaches the desired conditions for adjacent points. By this means one is in effect smoothing out those fluctuations in the data which are a result of random errors but retaining those fluctuations which are of a size large compared with the uncertainty and are therefore real. Although small kinks no doubt do occur in the number-density profile they can be resolved only if data of sufficient number and quality are obtained so that the uncertainties become smaller than the kinks.

The empirical procedure discussed above is clearly not uniquely defined and therefore requires some judgement in its application. The statement that one can average two points if they are close enough together in altitude so that the uncertainties are of the same order as or somewhat larger than the true difference in magnitude, is vague in the exact meaning of the words "somewhat larger than". It seems prudent therefore to be conservative in applying the averaging process. The important idea to be gained is that there are poor ways to handle data, there are relatively good ways to handle data, and there is the one correct (in a strict statistical sense) way to handle the data. The technique we have suggested would be classed as one of the relatively good ways. This approach can very substantially reduce the temperature uncertainty below that given by straight use of Equations (10), (11), (12), and (13). Now, in fact, as the number of data points increases indefinitely the uncertainty tends to zero as it should. The above discussion will be amplified in the next section, dealing with numerical evaluations of the error equations.

SECTION 5

ERROR ANALYSIS - NUMERICAL ILLUSTRATION

In order to make the preceding discussion as clear as possible, the basic results of an analysis of the uncertainties in a hypothetical, simplified experiment involving number-density data are presented. We assume that by means of a mass spectrometer, values of number density for helium and argon over an extended range have been obtained. For simplicity, the points are assumed to be at intervals of 10 geopotential kilometers beginning at 150 km' and extending to 700 km'. The values of the number density are assumed to follow the curves shown in Figure 1, and the uncertainty at each point is assumed to be given by Equation (16) with $\Delta n = 10^{10} \text{ meter}^{-3}$, and $\Delta t = .01$. This value of Δt assumes that the interpretation of the data is accurate. In many experiments where exactly what is being measured is open to question, the value of Δt may be much higher than 1%. However, this type of error is not statistical but rather systematic in nature and cannot be included in a statistical analysis.

At an altitude of 400 km' the number density of the argon is estimated to be $1.5 \times 10^{10} \text{ m}^{-3}$, which is approaching the limits of instrument sensitivity. Therefore, above 400 km' about all one can say from the

experiment is that the argon number density is less than the instrument sensitivity of $1 \times 10^{10} \text{ m}^{-3}$; there are no such restrictions on the helium data, however, since the helium density never falls below the instrument sensitivity in the altitude range considered.

It should be noted in the calculations to follow that since the data are assumed smooth, the computed temperature profile and the uncertainty profile are also smooth. In a real experiment, of course, the data points do not all lie on a smooth curve but instead fluctuate about an average curve. Thus, the computed temperatures and uncertainties will in a real experiment fluctuate with amplitudes consistent with the fluctuations in the number densities.

As a first step in our illustrative example, the uncertainties are computed assuming point to point integration without averaging any groups of points. The basic results are shown in Figure 6, in which the solid curves come from the double-gas equations, and the dashed curves come from the single-gas equations. An initial uncertainty $\delta T_{400} = 1000^\circ\text{K}$ was assumed for the heavy-gas-down calculation, and the resulting value $\delta T_{150} = 7.8^\circ\text{K}$ was then used in the light-gas-up calculation. It is seen that as expected, the conditions $(\delta T \frac{a}{b})_h \approx (\delta T \frac{a}{b})_d$ and $(\delta T \frac{a}{b})_\ell \approx (\delta T \frac{a}{b})_d$ hold very closely, so that for this experiment it makes little difference whether the double-gas or single-gas equations are used. It is interesting and very encouraging that the temperature uncertainty at 150 km' is as small as it is. The reason is that since the argon has a very high density in the interval 150 km' to 250 km' compared with the spectrometer

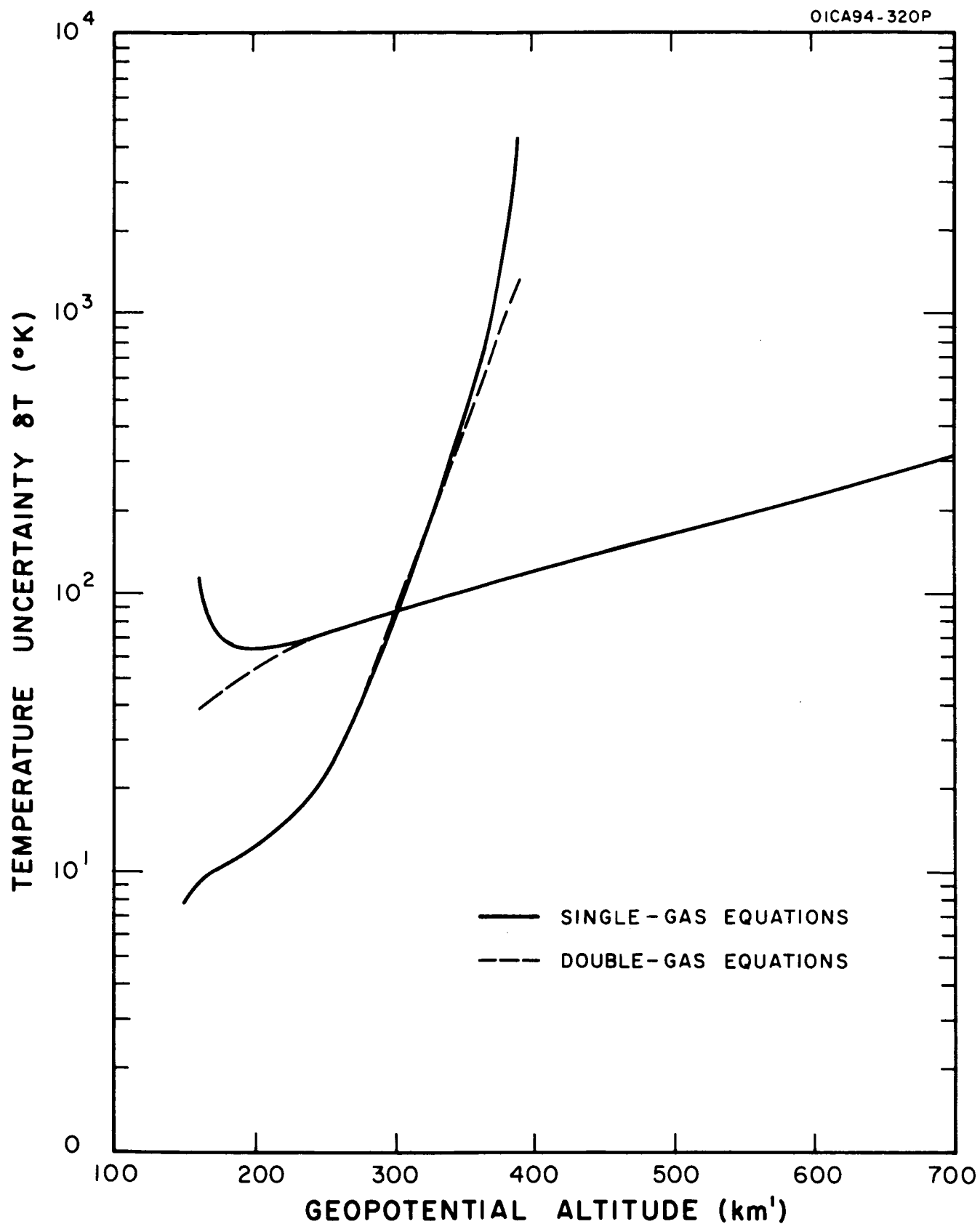


Figure 6. Temperature uncertainties computed by a point to point integration without averaging any group of points.

sensitivity, the uncertainties come mainly from the quantity Δt in Equation (16). The result $\delta T_{150} = 7.8^{\circ}\text{K}$ was obtained assuming $\Delta t = 1\%$, but even if $\Delta t = 5\%$ were used, one would still obtain tolerable uncertainties, namely, $\delta T_{150} \approx 40^{\circ}\text{K}$.

Since the argon number density decreases very rapidly with increasing altitude, the percent uncertainty in argon density increases very rapidly with altitude. On the other hand, since the helium densities decrease only moderately with increasing altitude, the percent uncertainty in helium density increases only moderately. The downward proceeding equations are affected mainly by the argon data, while the upward proceeding equations are affected mainly by the helium data, and therefore the general behavior of the results is readily understandable.

The size of the altitude range of data is also important, and in general the larger the total altitude interval used the lower the temperature uncertainty. The results for the double-gas-down computation for an interval 400 to 150 km' and for 300 to 150 km' are shown in Figure 7. Although the two curves converge in the region 150 km' it is obvious that the curve computed from the higher starting altitude always gives the lower uncertainty.

The uncertainties computed in the above treatment of the data can be significantly improved upon by using the data to better advantage. Following the procedure described in Section 4, the data are averaged consistent with the empirical (and rather conservative) requirement that

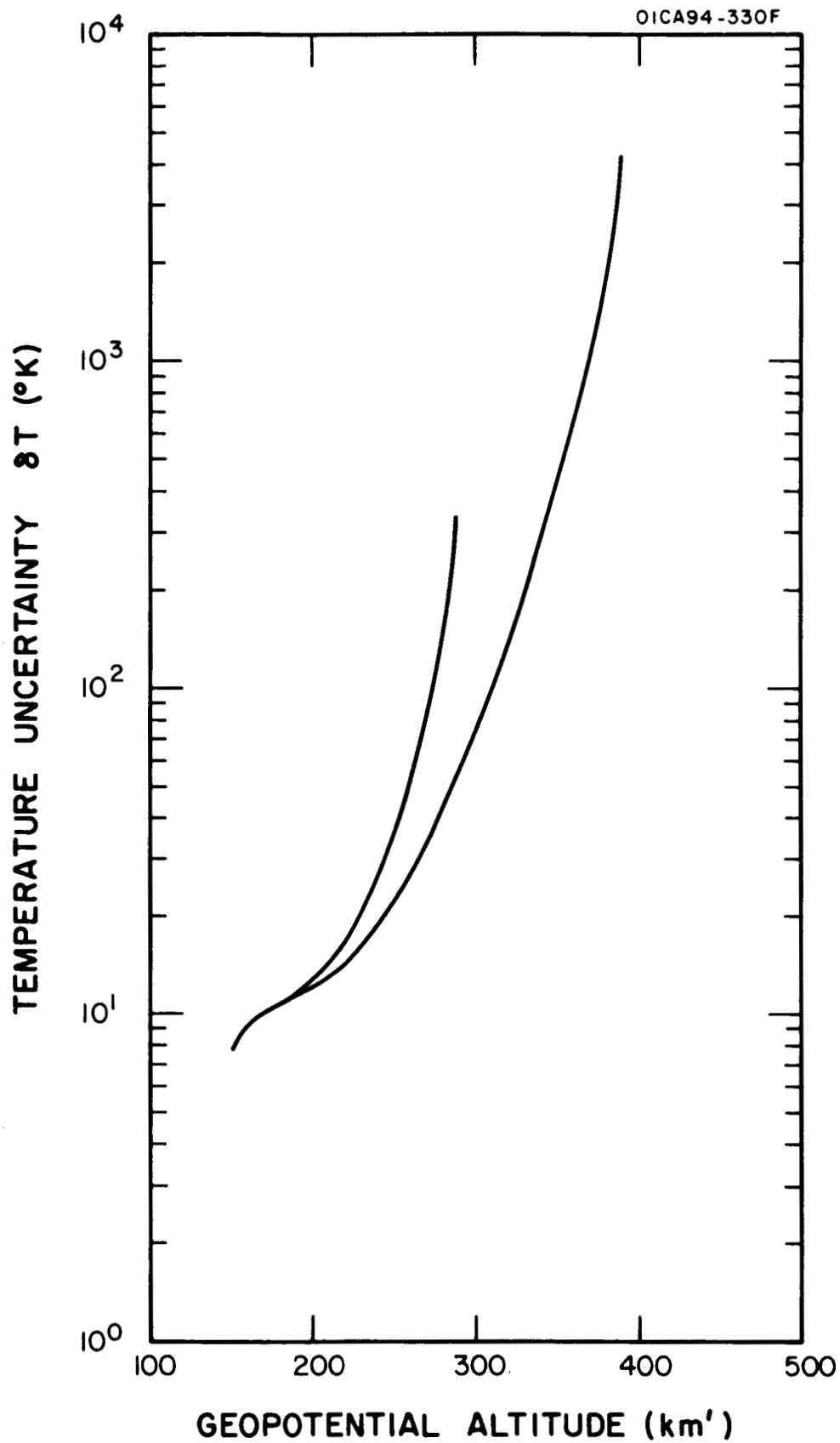


Figure 7. Temperature uncertainties computed by a point to point integration of the single-gas-down uncertainty equations without averaging any groups of points.

the new adjacent points, arrived at by averaging neighbors, have true differences greater than the new uncertainties. Thus the averaging process was carried out as follows:

	<u>Altitude Range of Argon (km')</u>	<u>Altitude Range of Helium (km')</u>
no averages taken	150 - 380	150 - 260
average groups of two	390 - 400	270 - 540
average groups of three	-----	550 - 660
average groups of four	-----	670 - 700

This averaging process produced a significant decrease in the overall uncertainties in the light gas results as shown in Figure 8. These are about 10 to 15% decrease for the two-point average 16% for the three-point average and 20% for the four-point average. The reduction of uncertainty produced by averaging of heavy gas data are not shown, because the large slope tends to mask the effect for the conditions assumed. Only the single-gas equations were used in the calculation but obviously the double-gas equations must give essentially the same results.

In Figure 9, the standard atmosphere temperature profile, with which our number densities are consistent, is shown along with the estimated uncertainties one would obtain in an experiment as outlined in the illustration above. This experiment is clearly not one in which good temperature results are obtained above 275 km', mainly because of the relatively sparse amount of assumed data. Consider what the results might

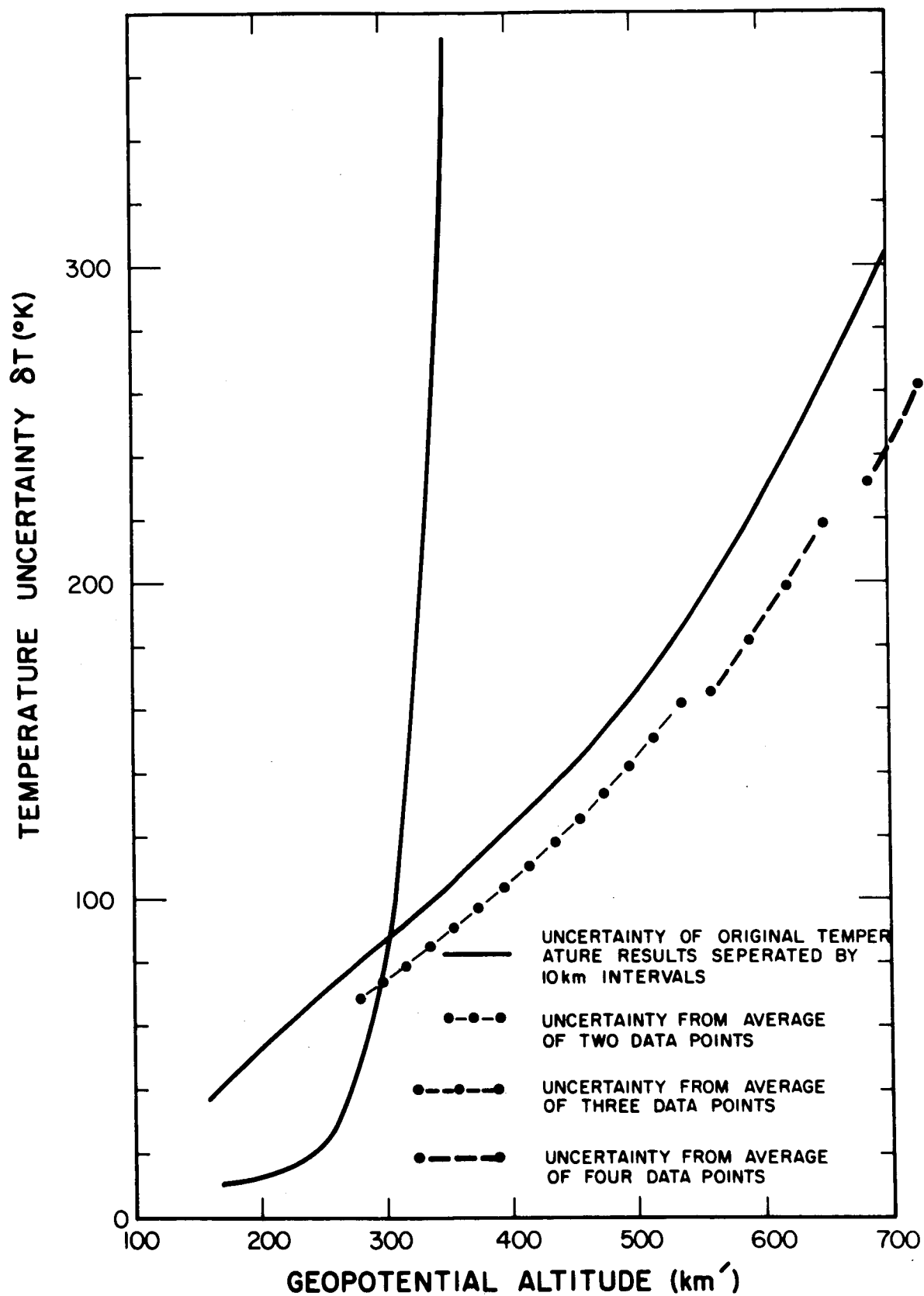


Figure 8. Temperature uncertainties computed after a simple averaging of small groups of data points.

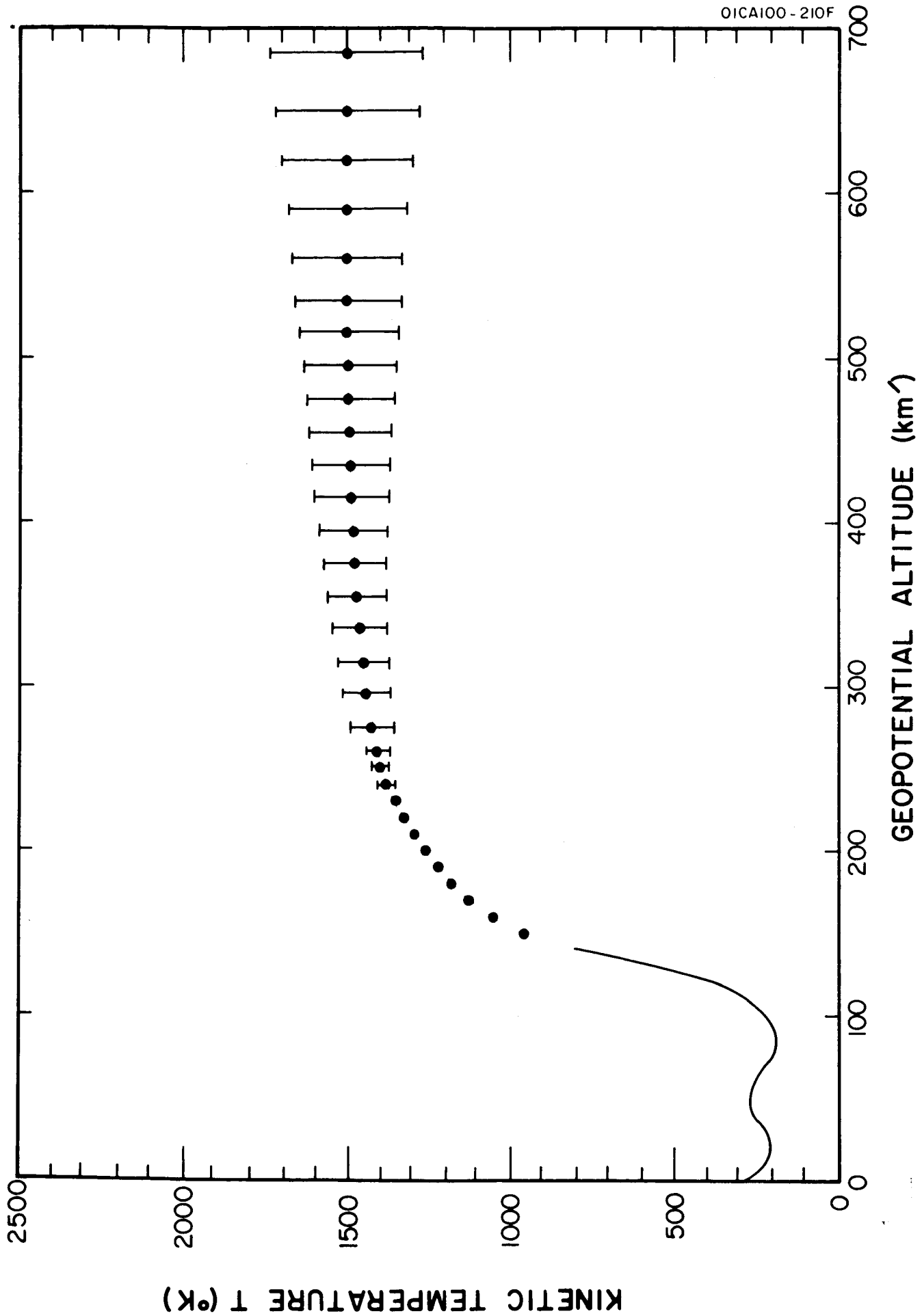


Figure 9. Uncertainties in the temperature-altitude profile deduced from assumed values of helium and argon number density data for 10 km³ increments in altitude using averaging techniques.

have been, had the data points been only 1 km' apart instead of 10 km' apart. The straight point-to-point calculation would improve the results only slightly compared with the 10 km' data. With the 1 km' data, however, the averaging process could very effectively be used to reduce errors by a factor of 2 or 3 over some of the altitude range. The reason for this is simply that as the number of points averaged together increases, the uncertainties δn and consequently δT decrease as the square root of the number of points averaged.

One can see that there are likely to be regions of altitude where it is not very important to get good data. For example, if one wants to get as good a value of temperature as possible at 150 km', the most important data is the argon number densities in the region 150 to 250 km'. Argon data above 250 km' does not have much influence on the calculated temperature at 150 km'. In the light-gas-up equation, however, improvements in the data taken at low altitudes is strongly felt at the higher altitudes.

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